The barotropic stability of the mean winds in the atmosphere

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This paper considers the stability of a barotropic current on a beta earth. The motion is assumed to be horizontal, non-divergent and barotropic. The current is taken to be of the form $U(y) = A\,\mathrm{sech}^2\,by + B$. The perturbations are required to approach zero as y approaches $\pm\infty$. We introduce the non-dimensional wave-number l and a parameter χ, which is a measure of the rotation effect. χ is inversely proportional to β.

There are only two kinds of perturbations: symmetric disturbances (those with maximum amplitude at $y = 0$) and antisymmetric disturbances (those with zero amplitude at $y = 0$). We find the neutral curve in the (χ, l^2)-plane for both types of disturbances. The rates of amplification in the immediate vicinity of the neutral curves are also found. It is seen that the beta effect, which is due to the earth's rotation, tends to stabilize the current. For the symmetric disturbances we find a band of unstable wavelengths when $\chi > \frac{1}{2}$; and for large χ the estimated curve of the maximum value of the imaginary part of the phase velocity is asymptotic to the lower branch of the neutral curve. The antisymmetric disturbances are more stable than the symmetric disturbances.

1. Introduction

The motivation for this investigation is the problem of the stability of the mean westerly current in the upper atmosphere. This current, which varies in strength with latitude and height, is strongest near the tropopause at about 30° N. (Mintz 1955). At present, a complete analysis of the three-dimensional stability problem is too complicated for mathematical treatment. In order to simplify the analysis, most investigators have considered one of two approaches. The first is to allow the current to vary in the vertical direction only and to neglect latitude variations. This formulation is known as the baroclinic stability problem (see Charney 1947; Kuo 1952). The other approach is to allow the current to vary in the latitudinal direction only, and to neglect vertical variations. This formulation is known as the barotropic stability problem. In both formulations frictional forces are neglected.

Here we consider the latter approach. The barotropic stability problem has been studied by Foote & Lin (1951), and Kuo (1949, 1951). These authors show that the barotropic basic current is stable if the absolute vorticity profile is monotonic. It will be seen in the analysis below that one of the primary effects of the earth's rotation is to reduce the instability of the barotropic current. This effect is the characteristic difference between the stability of the current considered below and the stability of the usual two-dimensional jet of hydrodynamics.

Earlier, Kuo (1949) considered a symmetric jet with an extremum of absolute vorticity on either side of the jet maximum. He finds a band of unstable wavelengths between the long neutral waves of Rossby *et al.* (1939) and Haurwitz (1940), and the shorter stable waves. The phase velocities of these waves are between the maximum and minimum velocity for the latitude belt. The long neutral waves all have phase velocities smaller than $U_{\min}$. In addition, Kuo proves that no neutral wave can have a phase velocity greater than $U_{\max}$.

In his second paper, Kuo (1951) finds the neutral-wave perturbation by numerical integration for a jet on a spherical earth. The wave-number (number of waves around the globe) of this wave is 9·5, and he infers that the maximum instability is at wave-number 4 or 5.

Here the work of Kuo (1949) is extended by considering a more realistic symmetric jet. The problem is non-dimensionalized, and the parameters l and χ are defined: l is the non-dimensional wave-number and χ is a measure of the rotation effect. χ is inversely proportional to the gradient of absolute vorticity of the earth's rotation. The stability of the jet is considered for all values of χ. Kuo's analysis is extended by considering both symmetric and antisymmetric disturbances. (Symmetric disturbances are those with maximum amplitude at the jet, and the antisymmetric disturbances are those with zero amplitude at the jet.) Kuo (1949, 1951) does not consider the antisymmetric disturbances. We find that the antisymmetric disturbances are more stable than the symmetric disturbances.

2. The perturbation equations and boundary conditions

The motion is assumed to be horizontal, non-divergent and barotropic. The basic motion consists of a fluid velocity from west to east. We assume that the largest velocities in the basic flow occur in a very narrow latitude belt, and that the basic velocity quickly approaches a constant value as we approach either the pole or the equator. Under these conditions it is legitimate to approximate the spherical co-ordinates on the earth by Cartesian co-ordinates, x, y and z, directed toward the east, north and vertical, respectively (see Long 1960). The respective velocities are u, v and w. The basic velocity takes the form $U = U(y)$. Since the flow is non-divergent and horizontal, we may define a stream-function for the perturbed motion by

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x}.$$

The dynamic equation to be satisfied for this motion is the two-dimensional vorticity equation. This equation states that for any fluid element the vertical component of absolute vorticity is conserved. The absolute vorticity includes both the relative vorticity due to motion relative to the earth and the vorticity of the earth's rotation. This equation takes the form

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\psi + (\beta - U'')\frac{\partial\psi}{\partial x} = 0. \tag{1}$$

Here a prime denotes a differentiation with respect to y, and $\beta = (d/dy)\,2\omega$, where ω is the vertical component of the earth's rotation. In the following analysis we may take $\beta =$ const., since we are considering a jet confined to a narrow latitude belt (see Long 1960). The approximation used above in which we replace the spherical co-ordinates by Cartesian co-ordinates, and consider β to be constant is known as the beta plane approximation in meteorological literature (see Rossby *et al.* 1939, and Haurwitz 1940).

If we now set $\psi(x,y,t) = e^{i\alpha(x-ct)}\phi(y)$, (1) becomes

$$\phi'' - \alpha^2\phi + \{(\beta - U'')/(U-c)\}\phi = 0, \tag{2}$$

where α is the wave-number and c is the phase velocity (which may be complex), i.e.

$$c = c_r + ic_i. \tag{3}$$

If $c_i \neq 0$, the stream function contains a term exponential in time; if $c_i > 0$, the wave is amplified; if $c_i < 0$, the wave is damped; and if $c_i = 0$, the wave is neutral.

The form of $U(y)$ is taken as

$$U(y) = A\,\mathrm{sech}^2\,by + B, \tag{4}$$

where A, B and b are arbitrary constants to be specified in any particular case. We now non-dimensionalize equation (2). For this purpose we define:

$$\left.\begin{aligned} x^* = bx, \quad y^* = by, \quad t^* = bAt, \quad c^* = (c-B)/A, \\ l = \alpha/b, \quad \chi = \tfrac{1}{3}Ab^2/\beta, \quad U^* = \mathrm{sech}^2\,y^*, \quad \phi^* = \phi b/A. \end{aligned}\right\} \tag{5}$$

Thus, without the asterisks, the non-dimensionalized form of equation (2) becomes

$$\phi'' - l^2\phi + \{(\tfrac{1}{3}\chi^{-1} - U'')/(U-c)\}\phi = 0. \tag{6}$$

The graph of the non-dimensionalized velocity profile is shown in figure 1. In figure 2 we have plotted some typical absolute vorticity profiles for $U = \mathrm{sech}^2\,y$. In this figure the absolute vorticity is given by

$$\zeta - \zeta_0 = \tfrac{1}{3}\chi^{-1}y - U', \tag{7}$$

where ζ_0 is the absolute vorticity at $y = 0$.

We impose the boundary conditions that $\phi = 0$ at $y = \pm\infty$. Since U is symmetric, it is evident that the two independent solutions may be taken as an even function and an odd function of y respectively. The former corresponds to a symmetric, and the latter to an antisymmetric solution. If ϕ_1 is the symmetric solution and ϕ_2 is the antisymmetric solution, the boundary conditions become

$$\phi_1'(0) = 0, \quad \phi_1(\infty) = 0; \tag{8a}$$

$$\phi_2(0) = 0, \quad \phi_2(\infty) = 0. \tag{8b}$$

Thus, at the jet, ϕ_2 has zero amplitude and ϕ_1 has a maximum in amplitude.

3. The neutral solutions

In this section we state some results that Kuo (1949) found concerning neutral solutions for his problem, and then we find the neutral solutions for the above velocity profile. For a finite jet confined between boundaries, Kuo (1949)

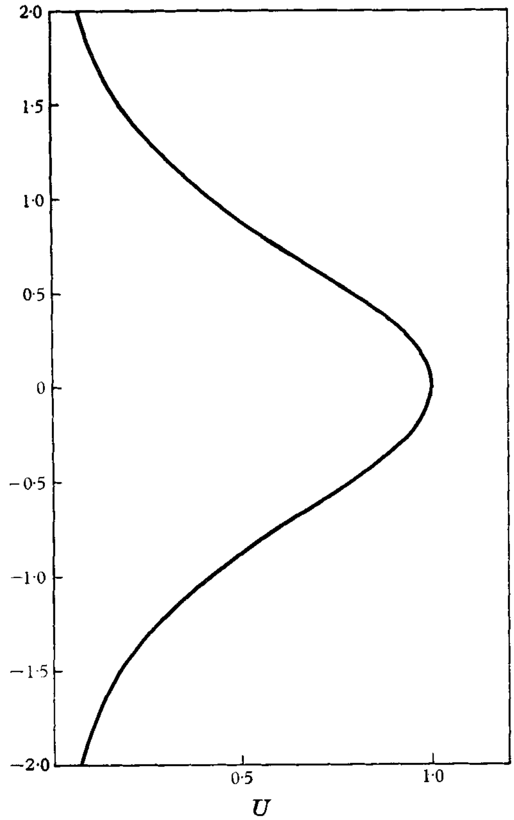

FIGURE 1. $U = \mathrm{sech}^2\, y$.

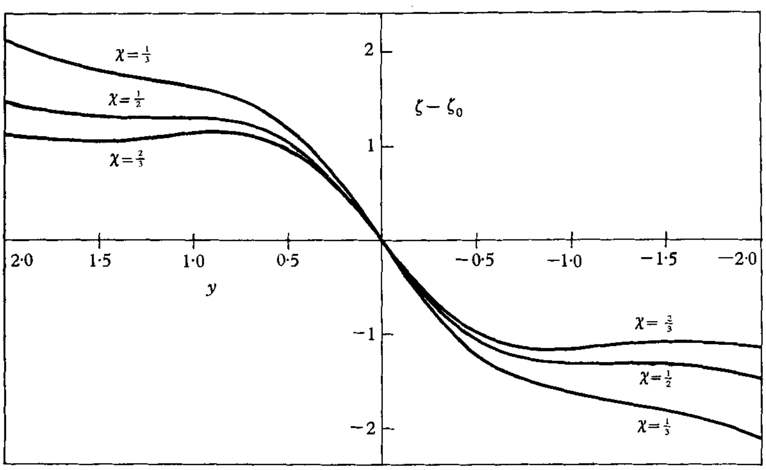

FIGURE 2. The absolute vorticity profiles associated with $U = \mathrm{sech}^2\, y$ for representative values of χ.

proves that if $U = c$ at some point $y = y_c$ for a neutral wave, then $\beta - U''$ is zero at that point. He also shows that no wave can move with a phase velocity greater than $U_{\max}$. These proofs, which are based on the Sturm comparison theorem (Ince 1944), can be carried over directly to our case with only small revisions. Thus we assume these theorems in the following work.

First, we find the phase velocities of the neutral waves for which $c > 0$ from the roots of the equation $\frac{1}{3}\chi^{-1} - U'' = 0$. By differentiation we find that

$$\tfrac{1}{3}\chi^{-1} - U'' = 6\,\mathrm{sech}^4 y - 4\,\mathrm{sech}^2 y + \tfrac{1}{3}\chi^{-1}$$

or
$$\tfrac{1}{3}\chi^{-1} - U'' = 6(U - c_1)(U - c_2), \tag{9}$$

where
$$c_1 = \tfrac{1}{3}\{1 + (1 - \tfrac{1}{2}\chi^{-1})^{\frac{1}{2}}\} \quad \text{and} \quad c_2 = \tfrac{1}{3}\{1 - (1 - \tfrac{1}{2}\chi^{-1})^{\frac{1}{2}}\}. \tag{10}$$

These are the phase velocities of the neutral waves for which $c > 0$.

The differential equation (6) becomes

$$\phi'' + [6(U - c_{1,2}) - l^2]\,\phi = 0, \tag{11}$$

where either subscript on c may be valid, depending on the value of the phase velocity. If the phase velocity is c_1, we have c_2 in (11), and vice versa. Equation (11) is of the form

$$\phi'' + [6\,\mathrm{sech}^2 y - k^2]\,\phi = 0, \tag{12}$$

where
$$k^2 = 6c_{1,2} + l^2. \tag{13}$$

We make a change of variable and set $Z = \tanh y$. The differential equation then becomes

$$(1 - Z^2)\frac{d^2\phi}{dZ^2} - 2Z\frac{d\phi}{dZ} + \left[6 - \frac{k^2}{1 - Z^2}\right]\phi = 0. \tag{14}$$

This equation is now compared to Legendre's equation

$$(1 - Z^2)\frac{d^2W}{dZ^2} - 2Z\frac{dW}{dZ} + \left[\nu(\nu+1) - \frac{k^2}{1 - Z^2}\right]\phi = 0. \tag{15}$$

For correspondence we need $\nu = 2$ or $\nu = -3$.

Equation (15) has been discussed fully in the Bateman Manuscript Project (1953) and in other references. The only solutions meeting the boundary conditions (8*a*) or (8*b*) are the associated Legendre polynomials $P_2^k(Z)$, where $k = 1$ or $k = 2$.

Thus, for boundary conditions (8*a*), we find the neutral solutions

$$\phi_1 = 1 - Z^2 = \mathrm{sech}^2 y, \tag{16}$$

provided $k = 2$, i.e.
$$4 = 2\{1 \pm (1 - \tfrac{1}{2}\chi^{-1})^{\frac{1}{2}}\} + l^2. \tag{17}$$

These are the only symmetric neutral solutions with $c > 0$.

Equation (17) determines l^2 as a function of χ for the neutral wave. l^2 is plotted against χ in figure 3. Because of the $\pm$ sign, there are two neutral waves for each χ if $\chi > \frac{1}{2}$. At $\chi = \frac{1}{2}$, there is only one neutral wave; and for $\chi < \frac{1}{2}$ there are no neutral waves with $c > 0$. Along the upper branch of the neutral curve in figure 3, $c = \frac{1}{3}\{1 + (1 - \frac{1}{2}\chi^{-1})^{\frac{1}{2}}\}$, and along the lower branch $c = \frac{1}{3}\{1 - (1 - \frac{1}{2}\chi^{-1})^{\frac{1}{2}}\}$. In the next section it is shown that waves are amplified for wavelengths between those of the two neutral waves. Outside this wavelength band there are stable waves.

For boundary conditions (8*b*), i.e. for antisymmetric solutions, we find for the unique neutral solution with $c > 0$:

$$\phi_2 = Z(1-Z^2)^{\frac{1}{2}} = \tanh y \operatorname{sech} y, \tag{18}$$

provided $k = 1$, i.e.

$$1 = 2\{1-(1-\tfrac{1}{2}\chi^{-1})^{\frac{1}{2}}\}+l^2. \tag{19}$$

Equation (19) has only a minus sign since $(1-\frac{1}{2}\chi^{-1})^{\frac{1}{2}}$ would imply a negative l^2. l^2 is plotted against χ in figure 4. In this case we have only one neutral wave for each χ if $\chi \geqslant \frac{2}{3}$. For this neutral wave $c = \frac{1}{3}\{1+(1-\frac{1}{2}\chi^{-1})^{\frac{1}{2}}\}$. As shown in the next section, the waves with wavelengths longer than this neutral wave are unstable and those with shorter wavelengths are stable.

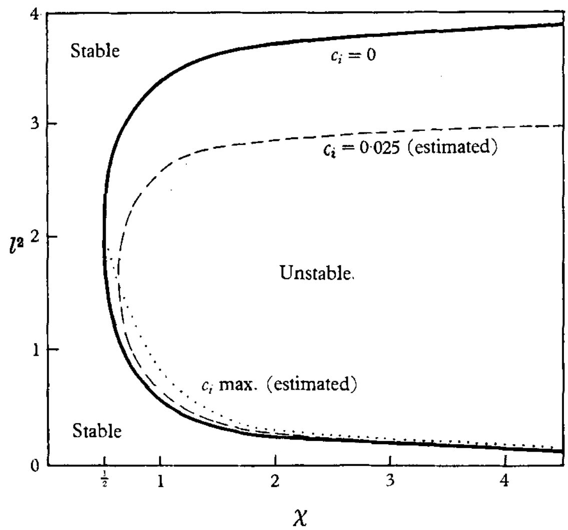

FIGURE 3. The stability of the symmetric disturbances ϕ_1.

In addition to these neutral waves, there is a group of neutral waves which are bounded as $y \to \pm\infty$, but which do not meet the boundary conditions (8*a*) or (8*b*). These waves all have $c < 0$ and correspond to the waves of Rossby *et al.* (1939) and Haurwitz (1940). It should be noted here that, in terms of dimensional quantities, $c < 0$ merely means that the dimensional phase velocity is less than U_∞; it does not necessarily mean that the waves propagate toward the west. From equation (16) we see that as $y \to \pm\infty$, $\phi \to \mathscr{A}\cos my + \mathscr{B}\sin my$. Also as $y \to \pm\infty$ we find that

$$l^2+m^2 = -1/3\chi c, \tag{20}$$

which is the frequency equation for these waves. This is the usual formula for the speed of Rossby waves. Kuo (1949) finds similar waves.

4. The amplified waves

To find c_i near the neutral curve in the (χ, l^2)-plane we may expand c in a Taylor series of the form:

$$c = c_0+\frac{\partial c}{\partial s}ds+\frac{\partial c}{\partial \chi^{-1}}d\chi^{-1}+\text{etc.}, \tag{21}$$

where $s = -l^2$ and c_0, $\partial c/\partial s$ and $\partial c/\partial \chi^{-1}$ are evaluated at some point (χ_0, l_0^2) on the neutral wave. In the following approach the derivatives $\partial c/\partial s$ and $\partial c/\partial \chi^{-1}$ are calculated from the neutral solution and the higher derivatives are neglected. Hence we should have a good approximation to c if the values of l^2 and χ are sufficiently close to the neutral curve.

To find† the derivative $\partial c/\partial s$ we take the derivative of (6) with respect to s:

$$\dot{\phi}'' + s\dot{\phi} + \frac{\frac{1}{3}\chi^{-1} - U''}{U-c}\dot{\phi} + \left(1 + \frac{\frac{1}{3}\chi^{-1} - U''}{(U-c)^2}\frac{\partial c}{\partial s}\right)\phi = 0, \tag{22}$$

where $\dot{\phi} = \partial\phi/\partial s$. If we multiply (22) by ϕ and (6) by $\dot{\phi}$, subtract, and then integrate, we find that

$$\frac{\partial c}{\partial s} = \int_0^\infty \phi^2 dy \Bigg/ \int_0^\infty \frac{U'' - \frac{1}{3}\chi^{-1}}{(U-c)^2}\phi^2 dy. \tag{23}$$

By a similar argument we find that

$$\frac{\partial c}{\partial \chi^{-1}} = \frac{1}{3}\int_0^\infty \frac{1}{U-c}\phi^2 dy \Bigg/ \int_0^\infty \frac{U'' - \frac{1}{3}\chi^{-1}}{(U-c)^2}\phi^2 dy. \tag{24}$$

In both these expressions we note that there is a singularity at $y = y_c$ where $U = c$ so that we must integrate around this point. To determine whether to take the path of integration above or below this point, we consider the viscous solution in the limit as the viscosity approaches zero. This problem has been investigated by Foote & Lin. They find that the path must be taken below the point $y = y_c$ if $U'(y_c) > 0$ and above this point if $U'(y_c) < 0$. Since $U'(y_c) < 0$ for $y > 0$, we must take the path of integration above $y = y_c$.

The expressions (23) and (24) can now be used to calculate the values of $\partial c/\partial s$ and $\partial c/\partial \chi^{-1}$ along the neutral curves. If we use the variable $Z = \tanh y$ and now integrate from $Z = 0$ to $Z = 1$, the values of $\partial c/\partial s$ and $\partial c/\partial \chi^{-1}$ can be found directly by integration. From the solution ϕ_1 on the neutral curve in figure 3 we find that

$$\frac{\partial c}{\partial s} = \frac{1}{3c_0[-6 + (1-3c_0)(1-c_0)^{-\frac{1}{2}}(\log\kappa + i\pi)]}, \tag{25}$$

where $$\kappa = \{1 + (1-c_0)^{\frac{1}{2}}\}/\{1 - (1-c_0)^{\frac{1}{2}}\}.$$

Likewise we find that

$$\frac{\partial c}{\partial \chi^{-1}} = \frac{2 + c_0(1-c_0)^{-\frac{1}{2}}(\log\kappa + i\pi)}{12c_0[-6 + (1-3c_0)(1-c_0)^{-\frac{1}{2}}(\log\kappa + i\pi)]}. \tag{26}$$

The factor of main interest is $\partial c_i/\partial s$. We find for this term

$$\frac{\partial c_i}{\partial s} = -\frac{\frac{1}{3}\pi c_0^{-1}(1-3c_0)(1-c_0)^{-\frac{1}{2}}}{[-6 + (1-3c_0)(1-c_0)^{-\frac{1}{2}}\log\kappa]^2 + [\pi(1-3c_0)(1-c_0)^{-\frac{1}{2}}]^2}. \tag{27}$$

Since we know $c_0 = \frac{1}{3}\{1 + (1-\frac{1}{2}\chi^{-1})^{\frac{1}{2}}\}$ along the upper neutral curve in the (χ, l^2) plane, we see that $\partial c_i/\partial s$ is positive there. Also, since $c_0 = \frac{1}{3}\{1 - (1-\frac{1}{2}\chi^{-1})^{\frac{1}{2}}\}$ along

† This calculation for $\partial c/\partial s$ follows that given by Lin (1953). Kuo does a similar calculation in his 1949 paper.

the lower neutral curve, we see that $\partial c_i/\partial s$ is negative there. Hence the region between the two neutral curves in figure 3 has amplified waves. Thus disturbances with wavelengths between those of the neutral waves are amplified and disturbances with wavelengths outside this band are stable.

Now consider the case of figure 4. Here we have only one neutral wave with $c > 0$ for $\chi \geqslant \frac{2}{3}$. This neutral wave has a velocity $c_0 = \frac{1}{3}\{1+(1-\frac{1}{2}\chi^{-1})^{\frac{1}{2}}\}$. We find, for the values of $\partial c/\partial s$ and $\partial c/\partial \chi^{-1}$,

$$\left.\begin{aligned}\frac{\partial c}{\partial s} &= \frac{1}{6[-1-2(1-3c_0)+(1-c_0)^{\frac{1}{2}}(1-3c_0)(\log\kappa+\pi i)]},\\ \frac{\partial c}{\partial \chi^{-1}} &= \frac{1}{12}\frac{\pi(1-c_0)^{\frac{1}{2}}(1-3c_0)}{[-1-2(1-3c_0)+(1-c_0)^{\frac{1}{2}}(1-3c_0)(\log\kappa+\pi i)]},\end{aligned}\right\} \tag{28}$$

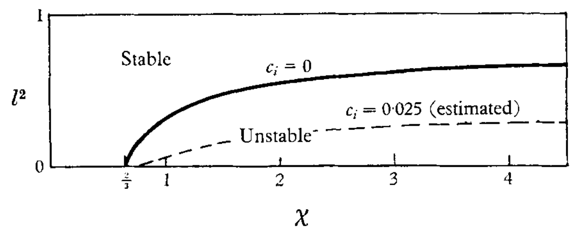

FIGURE 4. The stability of the antisymmetric disturbances ϕ_2.

and, for the value of $\partial c_i/\partial s$,

$$\frac{\partial c_i}{\partial s} = \frac{\pi(1-c_0)^{\frac{1}{2}}(1-3c_0)}{6[-1-2(1-3c_0)+(1-c_0)^{\frac{1}{2}}(1-3c_0)\log\kappa]^2+[\pi(1-c_0)^{\frac{1}{2}}(1-3c_0)]^2}. \tag{29}$$

Since $1-3c_0 < 0$ we find that $\partial c_i/\partial s$ is positive. Hence disturbances with wavelengths longer than the neutral wave are unstable and those with shorter wavelengths are stable.

In addition to the values of c_r and c_i calculated from $\partial c/\partial s$ and $\partial c/\partial \chi^{-1}$, we have the data of Lessen & Fox (1955) for the case $\chi = \infty$. They consider the case of an inviscid jet. Although they do not give a mathematical expression for $U(y)$, it is evident they used the form $\operatorname{sech}^2 y$ since this form of a velocity profile is the similarity solution for a laminar jet in a viscous fluid as given by Schlichting (1960). Furthermore, their graph of $U(y)$ agrees with our plot of $\operatorname{sech}^2 y$ to the order of the errors arising in reading the graph. They evaluate $\partial c/\partial s$ for the upper branch of the neutral curve at $\chi = \infty$ in figure 3 and $\partial c/\partial s$ for the neutral curve at $\chi = \infty$ in figure 4. To show how closely their results agree with the above theory, table 1 is given which compares these values for both formulations. It is seen that the difference in values is of the order of 0·3 %. In addition, Lessen & Fox calculate c_i and c_r *vs.* l^2 by numerical integration. For this work see figure 5.

We have given $\partial c_r/\partial s$, $\partial c_i/\partial s$ and $\partial c_i/\partial \chi^{-1}$ in table 2 for representative values of χ. From these values we estimate the curve $c_i = 0{\cdot}025$ for figure 3 and figure 4 where it is shown as a dashed line. Also the line of maximum c_i is estimated for figure 3 where it is shown as the dotted curve. We notice that this curve quickly becomes asymptotic to the lower neutral curve for large χ. This result is reflected in figure 5 (Lessen & Fox) where c_i appears to approach a maximum as l^2 approaches zero.

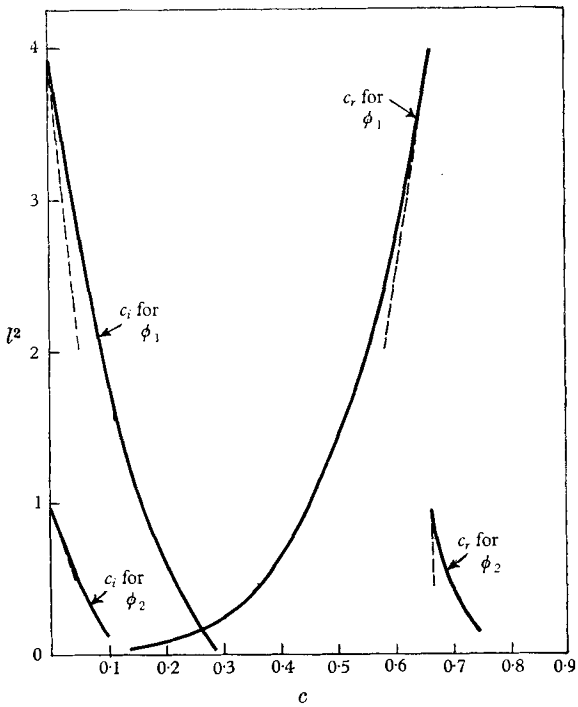

FIGURE 5. The data of Lessen & Fox (1955). The dashed lines are the slopes calculated from $\partial c_r/\partial s$ and $\partial c_i/\partial s$. The solid lines are the values found by numerical integration.

	Lessen & Fox	$U = \mathrm{sech}^2\, y$
ϕ_1	$\dfrac{\partial c}{\partial s} = -0{\cdot}0421 + 0{\cdot}02771i$	$\dfrac{\partial c}{\partial s} = -0{\cdot}04217 + 0{\cdot}02771i$
ϕ_2	$\dfrac{\partial c}{\partial s} = 0{\cdot}0119 + 0{\cdot}09021i$	$\dfrac{\partial c}{\partial s} = 0{\cdot}01193 + 0{\cdot}09031i$

TABLE 1. The values of $\partial c/\partial s$ at $\chi = \infty$ as given by Lessen & Fox and by the above theory for both types of disturbances, ϕ_1 and ϕ_2.

From figures 3 and 4 we see that the symmetric disturbances are stable if $\chi \leqslant \frac{1}{2}$ and the antisymmetric disturbances are stable if $\chi \leqslant \frac{2}{3}$. For larger values of χ it also appears that the symmetric disturbances are more unstable than the antisymmetric disturbances in the sense that they have larger amplification rates. From the values of $\partial c_i/\partial s$ for both types of disturbance for a given χ it appears that the maximum value of lc_i is largest for the symmetric disturbances. Thus, if the jet is unstable, the fastest growing disturbances apparently are the symmetric disturbances of medium wavelength.

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	Upper curve, figure 3	Lower curve, figure 3	Figure 4	
$(c_i)_s$	0	0	—	$\chi = \frac{1}{2}$
$(c_r)_s$	−0·16667	−0·1667	—	
$(c_i)_{\chi^{-1}}$	−0·05344	−0·05344	—	
$(c_i)_{\chi^{-1}}$	Not calc.	−0·09870	—	$\chi = 0{\cdot}54$
$(c_i)_s$	0·02292	−0·08668	—	$\chi = \frac{7}{12}$
$(c_r)_s$	−0·09886	−0·3114	—	
$(c_i)_{\chi^{-1}}$	−0·03032	−0·1147	—	
$(c_i)_s$	0·02578	−0·1599	0·1141	$\chi = \frac{2}{3}$
$(c_r)_s$	−0·08410	−0·4004	−0·06404	
$(c_i)_{\chi^{-1}}$	−0·02578	−0·1599	−0·1141	
$(c_i)_s$	0·02805	−0·4898	0·1091	$\chi = 1$
$(c_r)_s$	−0·06379	−0·6894	−0·02366	
$(c_i)_{\chi^{-1}}$	−0·01983	−0·3464	−0·07715	
$(c_i)_s$	0·02827	−1·7707	0·09960	$\chi = 2$
$(c_r)_s$	−0·05123	−1·2967	−0·001848	
$(c_i)_{\chi^{-1}}$	−0·01632	−1·0223	−0·05750	
$(c_i)_s$	0·02800	−6·1087	0·09393	$\chi = 5$
$(c_r)_s$	−0·04550	−1·6081	0·007106	
$(c_i)_{\chi^{-1}}$	−0·01476	−3·2196	−0·04951	
$(c_i)_s$	0·02771	$-\infty$	0·09031	$\chi = \infty$
$(c_r)_s$	−0·04217	∞	0·01193	
$(c_i)_{\chi^{-1}}$	−0·01386	$-\infty$	−0·04516	

TABLE 2. The values of $\partial c_i/\partial s$, $\partial c_r/\partial s$ and $\partial c_i/\partial \chi^{-1}$ calculated along the neutral curves in figures 3 and 4. Here $(c_i)_s = \partial c_i/\partial s$, $(c_r)_s = \partial c_r/\partial s$ and $(c_i)_{\chi^{-1}} = \partial c_i/\partial \chi^{-1}$.

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